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The Adjoint State Method for the Viscoelastic Wave Equation in the Velocity-stress Formulation

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SUMMARY

Viscous attenuation can have a strong impact on seismic wave propagation, but is rarely taken into account in full waveform inversion (FWI). In the time domain, when viscoelasticity is considered, the displacement formulation of the wave equation is usually used. However, the adjoint state equations are quite different for the velocity-stress formulation than for the displacement formulation. In this paper, we derive the adjoint state equations for the viscoelastic wave equation based on the velocity-stress formulation. Using a modified definition of the memory variables commonly found in the literature, we define a Lagrangian from which the adjoint state equations and the misfit gradient are derived. The resulting expressions are similar to the displacement formulation, but differ by the source term and by the wavefield cross-correlations giving the misfit gradient. To validate our results, the misfit gradient obtained by the adjoint state method is compared to the misfit gradient calculated by finite-difference for a simplified problem, giving an excellent agreement. In short, this work gives the right adjoint state equations for the velocity-stress formulation, which is commonly used for time-domain viscoelastic modeling. Further studies are required to evaluate the performance of this approach in real FWI viscoelastic experiments.

Introduction

Viscous attenuation can have a strong impact on seismic wave propagation. However, it is rarely taken into account in full waveform inversion (FWI), even though the theory was developed from the very beginning (Tarantola, 1988). A number of different approaches have been proposed for viscous FWI, most notably in the frequency domain (Song et al., 1995) or in the Laplace-Fourier domain (Kamei and Pratt, 2013). The simplicity of modelling viscous attenuation in the frequency domain is one of its main advantages over the time-domain; one only has to define complex velocities to implement an arbitrary attenuation profile in frequency (Toksöz and Johnston, 1981). In contrast, the time-domain approach usually requires the resolution of additional differential equations for memory variables (Carcione et al., 1988; Robertsson et al., 1994), and obtaining a desired attenuation profile in frequency is not straightforward (Blanch et al., 1995). However, the time-domain approach remains useful for large 3D models where the memory usage of the frequency approach is prohibitive, or when many frequencies are needed during inversion (Fichtner, 2011). The literature on viscous FWI in the time-domain is much more tenuous than in the frequency domain. Most authors like (Liao and McMechan, 1995; Causse et al., 1999; Bai et al., 2014) only consider the viscoacoustic case. When viscoelasticity is considered like in Tarantola (1988) or more recently Fichtner et al. (2006) and Askan et al. (2007), the displacement formulation of the wave equation is used. However, the adjoint state equations are quite different for the velocity-stress formulation, and their results cannot be used directly with the formulation of Robertsson et al. (1994).

In this paper, we derive the adjoint state equations for the viscoelastic wave equations based on the velocity-stress formulation. Our approach is inspired by the method of Castellanos et al. (2011) who derived the adjoint state equations for the elastic case. The main body of the paper focuses on the adjoint state equations derivation with the method proposed by Plessix (2006). The misfit gradient obtained by the adjoint state method is then compared to the misfit gradient calculated by finite-difference for a simplified problem.

Theory

The goal of full waveform inversion is to estimate the viscoelastic parameters of the ground $\boldsymbol{m} = (\rho, M, \mu, \tau_p, \tau_s)$ based on some records of the ground motion $\boldsymbol{d}_i$, usually in the form of particle velocities or pressure. This is performed by the minimization of a cost function, usually taken as the least-squares misfit of the raw seismic traces:

$$J(\boldsymbol{\phi};\boldsymbol{m}) = \frac{1}{2}\left(\boldsymbol{S}(\phi_i) - \boldsymbol{d_i}\right)^T \left(\boldsymbol{S}(\phi_i) - \boldsymbol{d_i}\right), \tag{1}$$

where $\boldsymbol{S}$ is the sampling operator and where the seismic wavefield is described by the state vector:

$$\boldsymbol{\phi} = \left(v_x, v_y, v_z, \sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{xz}, \sigma_{yz}, R_{xx}, R_{yy}, R_{zz}, R_{xy}, R_{xz}, R_{yz}\right)^T, \tag{2}$$

where v is the particle velocity, σ is the stress and R is the memory variable defined below. In this paper, the forward model is given by the viscoelastic wave equation using the generalized standard linear solid as described by Carcione et al. (1988) and Robertsson et al. (1994). We use the formulation corrected for the phase velocity similar to Bohlen (2002). In order to derive the adjoint state equations, an alternative form of the memory variables is used: the memory variable R is the integration in time of the memory variables used by the previous authors. In a matrix formulation, the forward model is given by:

$$F(\boldsymbol{\phi};\boldsymbol{m}) = \partial_{tt}\boldsymbol{A}\boldsymbol{\phi} + \partial_t \boldsymbol{B}\boldsymbol{\phi} - \boldsymbol{C}\boldsymbol{\phi} - \boldsymbol{s} = 0, \tag{3}$$

with the following matrices and parameters:

$$\boldsymbol{A} = \boldsymbol{0}_9 \bigoplus_{l=1}^{L} \tau_{\sigma l}\boldsymbol{I}_6, \qquad \boldsymbol{B} = \boldsymbol{I}_{9+6L}, \qquad \boldsymbol{C} = \begin{bmatrix} \boldsymbol{0}_3 & \frac{1}{\rho}\begin{bmatrix} \boldsymbol{D}_1 & \boldsymbol{D}_2 \end{bmatrix}_{1+L} \\ \begin{matrix} \lambda_e \boldsymbol{D}_3 + 2\mu_e \boldsymbol{D}_1 \\ \mu_e \boldsymbol{D}_2 \\ \begin{bmatrix} \lambda_v \boldsymbol{D}_3 + 2\mu_v \boldsymbol{D}_1 \\ \mu_v \boldsymbol{D}_2 \end{bmatrix}_L \end{matrix} & \boldsymbol{0}_{6+6L} \end{bmatrix}, \tag{4a}$$

$$\boldsymbol{D}_1 = \begin{bmatrix} \partial_x & 0 & 0 \\ 0 & \partial_y & 0 \\ 0 & 0 & \partial_z \end{bmatrix}, \qquad \boldsymbol{D}_2 = \begin{bmatrix} \partial_y & \partial_z & 0 \\ \partial_x & 0 & \partial_z \\ 0 & \partial_x & \partial_y \end{bmatrix}, \qquad \boldsymbol{D}_3 = \begin{bmatrix} \partial_x & \partial_y & \partial_z \\ \partial_x & \partial_y & \partial_z \\ \partial_x & \partial_y & \partial_z \end{bmatrix}, \tag{4b}$$

$$\lambda_e = M\frac{1+L\tau_p}{1+\alpha\tau_p} - 2\mu\frac{1+L\tau_s}{1+\alpha\tau_s}, \qquad \mu_e = \mu\frac{1+L\tau_s}{1+\alpha\tau_s}, \tag{4c}$$

$$\lambda_v = M\frac{\tau_p}{1+\alpha\tau_p} - 2\mu\frac{\tau_s}{1+\alpha\tau_s}, \qquad \mu_v = \mu\frac{\tau_s}{1+\alpha\tau_s}, \tag{4d}$$

$$\alpha = \sum_{l=1}^{L} \frac{\omega_0^2\tau_{\sigma l}^2}{1+\omega_0^2\tau_{\sigma l}^2}. \tag{4e}$$

The symbols $\boldsymbol{0}_n$ and $\boldsymbol{I}_n$ are the nxn zero and identity matrices and $\bigoplus$ stands for the direct sum. This formulation includes L Maxwell bodies, each with its own relaxation time $\tau_{\sigma l}$. The quality factor is a function of the attenuation levels τ_p and τ_s, and arbitrary attenuation profiles can be approximated by a superposition of Maxwell bodies (Bohlen, 2002).

Given the transform matrices:

$$\boldsymbol{\Lambda} = \rho\boldsymbol{I}_3 \bigoplus \frac{1}{3\lambda_e+2\mu_e}\boldsymbol{I}_1 \bigoplus \frac{1}{2\mu_e}\boldsymbol{I}_2 \bigoplus \frac{1}{\mu_e}\boldsymbol{I}_3 \bigoplus_{l=1}^{L} \left[\frac{1}{3\lambda_v+2\mu_v}\boldsymbol{I}_1 \bigoplus \frac{1}{2\mu_v}\boldsymbol{I}_2 \bigoplus \frac{1}{\mu_v}\boldsymbol{I}_3\right], \tag{5a}$$

$$\boldsymbol{T} = \boldsymbol{I}_3 \bigoplus_{l=1}^{L+1} \left[\boldsymbol{R} \bigoplus \boldsymbol{I}_3\right], \qquad \boldsymbol{R} = \begin{bmatrix} -1 & -1 & -1 \\ -1 & \frac{1}{2}\left(\sqrt{3}+1\right) & \frac{1}{2}\left(-\sqrt{3}+1\right) \\ -1 & \frac{1}{2}\left(-\sqrt{3}+1\right) & \frac{1}{2}\left(\sqrt{3}+1\right) \end{bmatrix}, \tag{5b}$$

a symmetric forward model is obtained by performing a change of variables and multiplying equation (3) by $\boldsymbol{\Lambda T}$, giving:

$$F'(\boldsymbol{\phi};\boldsymbol{m}) = \partial_{tt}\boldsymbol{A}'\boldsymbol{\phi}' + \partial_t\boldsymbol{B}'\boldsymbol{\phi}' - \boldsymbol{C}'\boldsymbol{\phi}' - \boldsymbol{s}' = 0, \tag{6}$$

with $\boldsymbol{A}' = \boldsymbol{\Lambda TAT}$, $\boldsymbol{B}' = \boldsymbol{\Lambda TBT}$, $\boldsymbol{C}' = \boldsymbol{\Lambda TCT}$, $\boldsymbol{s}' = \boldsymbol{\Lambda Ts}$ and $\boldsymbol{\phi}' = \boldsymbol{T\phi}$. With the transformed forward model, we define the following Lagrangian:

$$\mathfrak{L}\left(\tilde{\boldsymbol{\phi}}', \tilde{\boldsymbol{\psi}}'; \boldsymbol{m}\right) = J\left(\tilde{\boldsymbol{\phi}}'; \boldsymbol{m}\right) - \left\langle \tilde{\boldsymbol{\psi}}', \partial_{tt}\boldsymbol{A}'\tilde{\boldsymbol{\phi}}' + \partial_t\boldsymbol{B}'\tilde{\boldsymbol{\phi}}' - \boldsymbol{C}'\tilde{\boldsymbol{\phi}}' - \boldsymbol{s}' \right\rangle. \tag{7}$$

where $\tilde{\boldsymbol{\phi}}'$ is any realisation of the state vector and $\tilde{\boldsymbol{\psi}}'$ is any realisation of the adjoint state. The notation $\langle \boldsymbol{a}, \boldsymbol{b} \rangle = \int_T \int_X a_\alpha b_\alpha d\boldsymbol{x}dt$ denotes the scalar product. We transform the Lagrangian by integrating by part twice the term containing the matrix $\boldsymbol{A}'$, once the matrix $\boldsymbol{B}'$ and using Gauss theorem with the term containing the matrix $\boldsymbol{C}'$. Using the null boundary conditions and the symmetry of the three matrices, we can rewrite equation (7) as:

$$\mathfrak{L}\left(\tilde{\boldsymbol{\phi}}', \tilde{\boldsymbol{\psi}}'; \boldsymbol{m}\right) = J\left(\tilde{\boldsymbol{\phi}}'; \boldsymbol{m}\right) - \left\langle \partial_{tt}\boldsymbol{A}'\tilde{\boldsymbol{\psi}}' - \partial_t\boldsymbol{B}'\tilde{\boldsymbol{\psi}}' + \boldsymbol{C}'\tilde{\boldsymbol{\psi}}', \tilde{\boldsymbol{\phi}}' \right\rangle + \left\langle \tilde{\boldsymbol{\psi}}', \boldsymbol{s}' \right\rangle. \tag{8}$$

Equating to zero the derivative of equation (8) with respect to $\tilde{\boldsymbol{\phi}}'$ gives the adjoint state equations. Performing the back transformation, we obtain:

$$\overleftarrow{F}(\boldsymbol{\psi}, \boldsymbol{\phi}; \boldsymbol{m}) = \partial_{tt}\boldsymbol{A\psi} - \partial_t\boldsymbol{B\psi} + \boldsymbol{C\psi} - \boldsymbol{\Lambda}^{-1}\boldsymbol{T}\frac{\partial J}{\partial \boldsymbol{\phi}}\boldsymbol{T} = 0 \tag{9}$$

This last equation is similar to the usual back propagation equation obtained by Tarantola (1988) for the displacement formulation. The adjoint state equations for the velocity-stress formulation given here differ essentially by the source term. Its interpretation remains, however, the same: the adjoint state equations are the back-propagation in time of the data residuals. It thus can be obtained by the same modelling algorithm as the forward solution.

Finally, the misfit gradient for the viscoelastic parameters is obtained by the derivative of the Lagrangian with respect to the inversion parameters:

$$\frac{\partial J}{\partial m_\alpha} = \frac{\partial \mathfrak{L}}{\partial m_\alpha} = -\left\langle \boldsymbol{T}\boldsymbol{\psi}, \frac{\partial \boldsymbol{\Lambda}}{\partial m_\alpha} \boldsymbol{T} \left(\partial_{tt}\boldsymbol{A}\boldsymbol{\phi} + \partial_t \boldsymbol{B}\boldsymbol{\phi} - \boldsymbol{s}\right) \right\rangle . \tag{10}$$

The misfit gradient can thus be obtained by the cross-correlation of the forward and adjoint wavefields. This cross-correlation can be computed in time or in the frequency domain using Parceval's theorem.

Numerical validation

To test the validity of the misfit gradient obtained by the adjoint state equations, a synthetic 2D cross-well tomography survey is simulated. As no analytical solution exists for the misfit gradient, the adjoint state gradient is compared to the gradient computed by finite differences. The wells separation is 250 m and the source and receiver spacing are respectively 60 m and 12 m. Circular perturbations of 60 m radius for the five viscoelastic parameters were considered at five different locations over a homogeneous background with $V_p = 3500$ m/s, $V_s = 2000$ m/s, $\rho = 2000$ kg/m^3 and $\tau_p = \tau_s = 0.2$. The perturbations strength is 5 % of the constant value. Because significant crosstalk can exist between parameters (Kamei and Pratt, 2013), we computed the gradient for one type of perturbation at a time. For example, the *P*-wave velocity gradient is computed with constants models for all other parameters other than V_p. This eliminates any crosstalk and allows a better appraisal of the match between the gradient update and the given perturbations. The FD solution was obtained by perturbing each parameter of the grid sequentially by 2%, for all the grid position between the two wells.

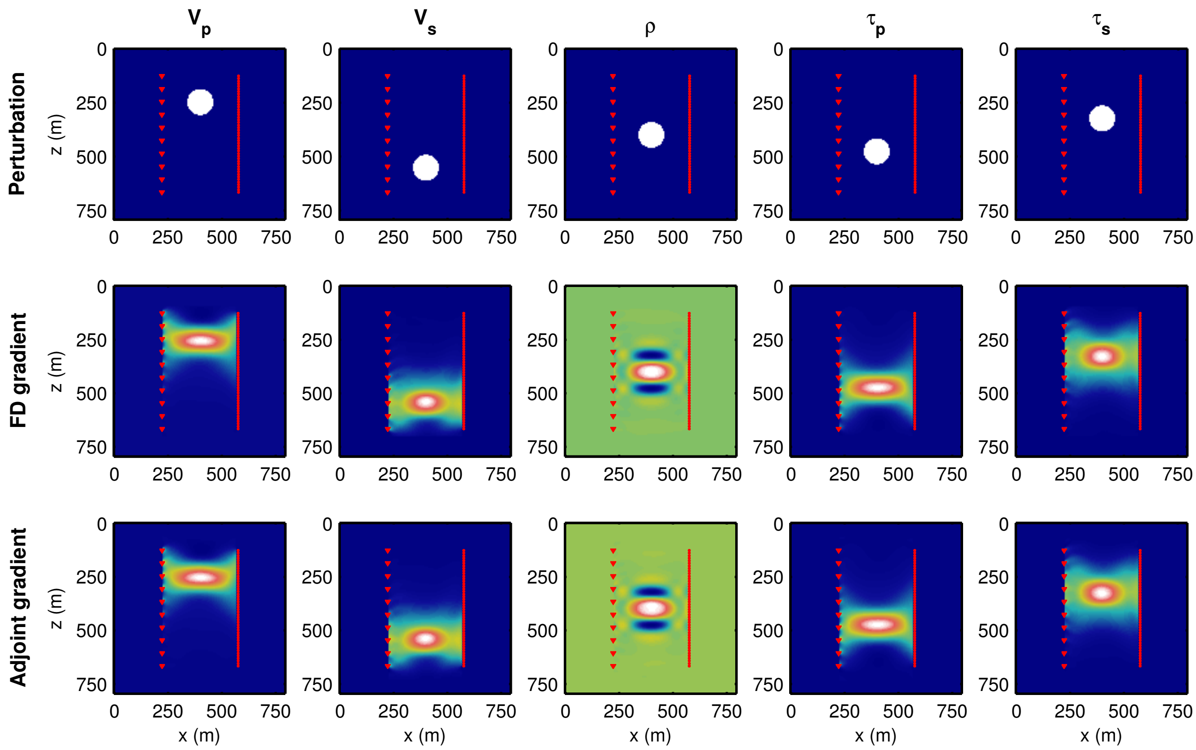

Figure 1 *A cross-well experiment to test the validity of the misfit gradient given by the adjoint state method. The red triangles represent the sources position and the red dots the receivers position. The color map is normalized for each panel*

In Figure 1, each column represents a different parameter. The first row shows the perturbation, the second row shows the misfit gradient obtained by finite differences and the third row shows the misfit gradient by the adjoint state equations. The excellent agreement between the adjoint state gradient and the FD gradient for all parameters can be visually appraised in Figure 1. In addition, the gradient correction is well centered on the perturbations. Hence, the inversion should converge to the right solution in the five different cases, the gradient pointing already toward the right direction. This is expected considering the small value of the perturbation used in this experiment.

Conclusions

In this paper, we derived the time-domain adjoint state equations for the viscoelastic wave equation in the velocity-stress formulation. The resulting equations differ from the displacement formulation by the source term and the cross-correlation expressions for the misfit gradient. We showed that the misfit gradient obtained by the adjoint state method is nearly identical to the misfit gradient obtained by finite differences. Further studies are required to assess the potential of this formulation to be used in real FWI cases. Also, the use of the attenuation parameters instead of the Q factors should be studied in more details.

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References

Askan, A., Akcelik, V., Bielak, J. and Ghattas, O. [2007] Full Waveform Inversion for Seismic Velocity and Anelastic Losses in Heterogeneous Structures. *Bulletin of the Seismological Society of America*, **97**(6), 1990–2008.

Bai, J., Yingst, D., Bloor, R. and Leveille, J. [2014] Viscoacoustic waveform inversion of velocity structures in the time domain. *Geophysics*, **79**(3), R103–R119.

Blanch, J.O., Robertsson, J.O.A. and Symes, W.W. [1995] Modeling of a constantQ: Methodology and algorithm for an efficient and optimally inexpensive viscoelastic technique. *Geophysics*, **60**(1), 176–184.

Bohlen, T. [2002] Parallel 3-D viscoelastic finite difference seismic modelling. *Computers and Geosciences*, **28**(8), 887–899.

Carcione, J.M., Kosloff, D. and Kosloff, R. [1988] Viscoacoustic wave propagation simulation in the earth. *Geophysics*, **53**(6), 769–777.

Castellanos, C., Etienne, V., Hu, G., Operto, S., Brossier, R. and Virieux, J. [2011] Algorithmic and methodological developments towards full waveform inversion in 3D elastic media. *Expanded Abstracts of 81st Annual Internat SEG Mtg*, 2793–2798.

Causse, E., Mittet, R. and Ursin, B. [1999] Preconditioning of full-waveform inversion in viscoacoustic media. *Geophysics*, **64**(1), 130–145.

Fichtner, A. [2011] *Full Seismic Waveform Modelling and Inversion*.

Fichtner, A., Bunge, H.P. and Igel, H. [2006] The adjoint method in seismology - I. Theory. *Physics of the Earth and Planetary Interiors*, **157**(1-2), 86–104.

Kamei, R. and Pratt, R.G. [2013] Inversion strategies for visco-acoustic waveform inversion. *Geophysical Journal International*, **194**(2), 859–884.

Liao, O. and McMechan, G.A. [1995] 2.5D full-wavefield viscoacoustic inversion. *Geophysical Prospecting*, **43**(8), 1043–1059.

Plessix, R.E. [2006] A review of the adjoint-state method for computing the gradient of a functional with geophysical applications. *Geophysical Journal International*, **167**(2), 495–503.

Robertsson, J.O.A., Blanch, J.O. and Symes, W.W. [1994] Viscoelastic finite-difference modeling. *Geophysics*, **59**(9), 1444–1456.

Song, Z., Williamson, P.R. and Pratt, R.G. [1995] Frequency-domain acoustic-wave modeling and inversion of crosshole data: Part II-Inversion method, synthetic experiments and real-data results. *Geophysics*, **60**(3), 796–809.

Tarantola, A. [1988] Theoretical background for the inversion of seismic waveforms including elasticity and attenuation. *Pure and Applied Geophysics*, **128**(1-2), 365–399.

Toksöz, M.N. and Johnston, D.H. [1981] *Seismic wave attenuation*. Soc of Exploration Geophysicists.